SEQUENTIAL SUBOPTIMAL ADAPTIVE CONTROL OF NONLINEAR SYSTEMS

By

THOMAS WALTER ELLIS

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Thomas Walter Ellis

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Chairman: Dr. A. P. Sage
Major Department: Electrical Engineering

In this dissertation two methods of sequential suboptimal adaptive control are presented which encompass both identification and control. A generally nonlinear differential system is modeled by a linear time-varying system of assumed form and of possibly lower dimension. This system is assumed stationary over subintervals of time which allows a controller to generate a sequential control law which minimizes a quadratic performance index. The use of a linear system model and a quadratic performance index allows the controller to operate in an on-line fashion due to the speed and ease at which the control can be calculated and applied.

The first control philosophy presented is a form of regulator control which enables the system to adapt to new trajectories as the system undergoes modifications, or is affected by noise or environmental changes. The given time interval of interest, $t\epsilon(t_o,t_f)$, is divided into N subintervals, $t\epsilon(t_i,t_{i+1})$, with $i = 0,1,\ldots,N-1$, and a system model is determined at each $t = t_i$. Then a suitable control

is found by the maximum principle which minimizes an integral of time weighted quadratic form of error and control effort over the time interval $t \epsilon (t_i, t_f)$.

The second control philosophy is a form of trajectory control which forces the system to track a predetermined desired trajectory. Again the time interval of interest, $t \epsilon (t_o, t_f)$, is divided into N subintervals, and a system model is determined at each $t = t_i$. The maximum principle is then utilized to determine the control which will drive the system sufficiently close to the desired trajectory at $t = t_{i+1}$.

Alternate means are given for the choice of a proper model, and invariant imbedding is presented as a means for parameter identification and state estimation which is particularly suited for on-line use. It is assumed that the problems of parameter identification, state estimation, and control may be decoupled.

Several applications of these techniques are given. The control of a nuclear reactor during startup in the presence of input and output noise is presented, along with a suboptimal guidance and control scheme for the low thrust orbital transfer problem which attempts to minimize fuel consumption, also in the presence of noise. Finally, the control of a nuclear rocket engine during startup is given along with reactivity profiles for several desirable temperature trajectories.

CHAPTER I

INTRODUCTION

Much recent attention has been given to the solution of optimal control problems for nonlinear systems. This effort has resulted in a variety of methods for the computational solution of nonlinear two-point boundary-value problems [1,2]. In these boundary-value problems half of the boundary conditions are specified at the final time. This implies that an a priori knowledge of the complete system dynamics must be known over the time interval of operation $t \epsilon (t_o, t_f)$. Thus, in a large number of cases, solution of an optimal nonlinear control problem results in the determination of an open loop control for a system with known dynamics over the time interval of operation. In many instances, a closed loop control is desired. Also, if there are process variations, environmental changes, or uncertainties in the system model, the complete knowledge of system dynamics necessary to predetermine the open loop control cannot be obtained. Furthermore, the presence of noise can possibly reduce the effectiveness of a predetermined control. For the case of a deterministic linear system with known constant coefficients and a quadratic cost function the closed loop control can be obtained with relative ease [3].

Pearson [4] recognizes this situation and proposes a closed loop suboptimal controller for nonlinear systems. It is derived from

the stable steady-state solution of the Ricatti equation which results from ordinary variational techniques. The nonlinear and nonstationary system is optimized with respect to a quadratic performance index by treating it as an instantaneously linear stationary system. Although the method has some merit, it has some obvious drawbacks. Namely, to generate a constant feedback controller, the time interval of interest must be large compared with system time constants. Also, in solving for the stable steady-state solution of the Ricatti equation, it is possible that existence and/or uniqueness difficulties can arise [5]. Finally, it can be shown that the success of the method depends strongly on the particular system, and if the linearized model varies radically, the results can be poor.

Kishi [6] has attempted to develop an on-line control scheme for linear systems. The time interval of interest is divided into subintervals and at each of these subintervals an open loop control is calculated by minimizing the performance index over the subinterval. This is similar to what Sage and Eisenberg [7] show although in this case nonlinear systems are treated, and the open loop control calculated at each subinterval is based on minimizing the performance index over the remaining time to go. Each of these methods works well for systems with nonlinearities in which the control has little effect, but in general, they would not be satisfactory, particularly where measurement noise is present.

This dissertation attempts to develop and provide experimental justification for an alternate approach to the sequential adaptive

control of noisy nonlinear systems based, in part, on the identification of a linear model and use of the real time computational simplicity of linear systems with quadratic cost functions. To be more specific, an attempt is made to develop a means of on-line control which can be computed rapidly due to the identification of a linear model for the plant, and which introduces feedback by sequentially monitoring the system at discrete time instants and updating the control. The model chosen is a linear time-varying system which is assumed stationary over subintervals of time, thus allowing a controller to generate a sequential control law which minimizes, not the given performance index, but a closely related one. The resulting control is of course only an approximation to the predetermined optimum control. However, due to noise, and process and environmental changes which cannot be foreseen, it may substantially reduce the cost over that which results from a predetermined open loop control. Even though the adoption of a suboptimal policy may result at times in a slight reduction in system performance, it is important to realize that perhaps the simplified calculation and utilization of the control might result in the suboptimal policy being optimal in a certain enlarged performance index.

The selection of a proper model is very important, and Chapter II outlines alternate means for its choice. Invarient imbedding is introduced as a means for parameter identification and state estimation which is particularly suited for on-line use [8]. It is assumed that the problems of parameter identification, state estimation, and control may be decoupled, although this does not result in a truly optimal system as shown by Sworder [9].

In Chapters III and IV, two methods for computing on-line control are presented. The first method enables the system to adapt to new trajectories as the system undergoes modifications. The second method attempts to keep the system tracking a precalculated trajectory. The choice of methods depends upon the particular situation and the type of control desired.

In Chapter V, applications of the above methods are given. Specifically, the startup of a nuclear reactor, the startup of a nuclear rocket engine, and a low thrust orbital transfer problem are given. These examples were chosen since they are of practical importance and current interest.

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CHAPTER II

MODELS, IDENTIFICATION AND STATE ESTIMATION

In optimal control theory it is desired to minimize a given cost function or performance index while controlling a system whose dynamic characteristics are given in differential or difference form relating the state variables and control variables. For systems whose dynamics are deterministic and completely known, the necessary conditions for optimal performance can be established in a formal manner once a cost function is specified. However, in many practical situations the dynamics of the system may be complex and vary in an unpredictable fashion. Thus it is necessary to incorporate an identification scheme which will sequentially update information on the system dynamics. Furthermore, in some cases either the mathematical description of the system dynamics may be unknown or perhaps the known system dynamics are of such complexity as to prohibit on-line computation. Then it is often convenient to use a model of assumed form which may be less complicated than the actual system.

This chapter attempts to present methods for choosing a satisfactory model along with an appropriate identification scheme which is suitable for use in a stochastic environment. The presence of noise demands a state estimation procedure, which, as shown later in the chapter, can be combined with the identification process.

By using this method it is implied that the estimation of the state variables and unknown parameters is separated from the derivation of the control. That is, once the model is identified and the state of the system is estimated, the information obtained is used to derive the appropriate control law. This type of approach, as shown in Figure 1, is referred to as an "ideal adaptive" system by Kalman [1].

It is known that if the controller and state estimator are optimized independently for a linear system with white Gaussian disturbances, an overall optimum system results for a quadratic performance index [2]. However, this is not true for nonlinear systems; that is, when estimates are used for the state variables in nonlinear systems, the overall system will not be optimal. Furthermore, Sworder [3] states that the operations of parameter estimation and optimization cannot, in general, be separated even for linear systems with a quadratic performance index. This is due to the inevitable coupling which exists between the unknown parameters and the control law. Since the combination of instantaneously best control, best state estimation, and best parameter estimation belongs to a class of extremely difficult unsolved problems, there is no alternative except to use the estimates which are available for deriving the control.

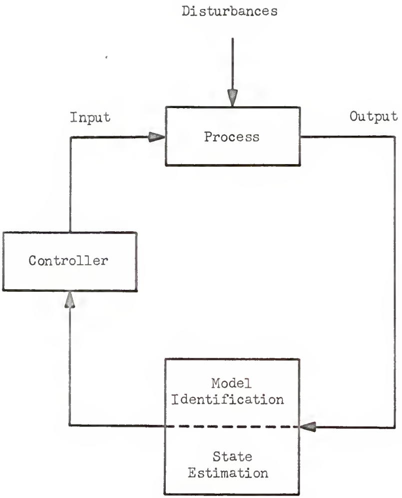

Figure 1. Block diagram of an "ideal adaptive" system

Models

When attempting to control a nonlinear time-varying system in some optimum fashion, it is always necessary to solve the inevitable two-point boundary-value problem which arises. Although many methods have been devised for its solution, all of them are iterative methods, or search methods, which consume much computation time. Also, convergence difficulties may prohibit satisfactory results, particularly if the system to be controlled is of high order or very nonlinear. Thus, it is often impossible to attempt on-line control of these systems.

When the system of interest is linear time-invariant many of the problems associated with computing the optimal control are alleviated. Therefore, when it is necessary to control a nonlinear adaptive system in an on-line fashion, it is advantageous to use a linear time-invariant model whenever possible.

Suppose it is desired to control a system which is assumed to be adequately described by

$$\begin{aligned} \dot{\underline{x}} &= \underline{f}\,(\underline{x}, \underline{u}, \underline{a}, t) \\ \dot{\underline{a}} &= \underline{0}, \ \underline{x}\,(t_o) = \underline{x}_o. \end{aligned} \tag{2.1}$$

These vector differential equations express the relationship between the state $\underline{x}$, an unknown constant parameter vector $\underline{a}$, the control vector $\underline{u}$ and time t. It is also desired to approximate (2.1) in a fashion such that an on-line solution for the control vector can be obtained. Several subclasses of the identification problem can now be posed.

If the system to be controlled possesses known dynamics and the control enters in a linear fashion (2.1) is the special form

$$\dot{\underline{x}} = \underline{g}(\underline{x},t) + H(\underline{x},t)\underline{u}(t). \tag{2.2}$$

It is then often possible to adequately approximate the system dynamics (2.2) by

$$\dot{\underline{x}} = A[\underline{x}(t_i),t_i]\underline{x}(t) + B[\underline{x}(t_i),t_i]\underline{u}(t) \tag{2.3}$$

for $t\epsilon(t_i,t_{i+1})$. In this case identification is accomplished by measurement of the state variables and computation of the A and B matrices of (2.3) [4,5]. If measurement noise is present, of course, it is necessary to filter the data in order to obtain a best estimate $\hat{\underline{x}}(t)$ of $\underline{x}(t)$. If the control does not enter linearly, or, if the system dynamics or system model are not precisely known, the above method of identification and modeling will normally not be satisfactory. In addition, even if the system dynamics can be represented as in (2.2), on-line computation of the control vector may be impossible if the dimensionality of the state vector is high. In all of these cases, identification of an approximate system model by other than direct measurement of the state vector at time t is desirable.

For the case where the model is to be of the same dimensionality as the state vector, (2.1) is identified in the form

$$\dot{\underline{x}} = A(t_i)\underline{x} + B(t_i)\underline{u} \tag{2.4}$$

for $t\epsilon(t_i,t_{i+1})$ where the A and B matrices are identified by some convenient on-line technique.

In the case where the number of state variables is too large to permit on-line control vector computation, a model of lower dimensionality than the original system (2.1) is identified. Specifically, the model identified is

$$\dot{\underline{y}} = A(t_i)\underline{y} + B(t_i)\underline{u} \tag{2.5}$$

for $t\epsilon(t_i, t_{i+1})$ where $\underline{y}(t) = h[\underline{x}(t)] \cdot \underline{y}(t)$ is of lower dimension than $\underline{x}(t)$ and the A and B matrices are identified as functions of the control variable and the new state variable.

It should be stressed that the choice of models is very important and should thus be chosen judiciously. Before starting an analysis, it is essential to examine most carefully the assumptions that are being made and the results expected. Quite often a simple model should be chosen initially and as more is learned about the system and its behavior, the model can be modified. Although the model may be an approximation, it may prevent more drastic approximations being made in the solution of the associated equations [6].

Identification and State Estimation

Parameter and state estimation may both be considered in the realm of the process of estimation, the process of making a decision, or judgment, concerning the approximate value of certain undefined objects when the decision is weighted, or influenced, by all available information. Most methods which have been proposed for state and parameter estimation have a common failing in that they are not generally

suited for on-line computation. One exception to this is the sequential estimation scheme developed by Detchmendy and Sridhar [7] and presented in this chapter. This scheme is an extension of earlier work done by Bellman and Kalaba [8,9] and has been modified for use with discrete systems by Sage and Masters [10,11]. It is particularly suited for on-line computation and also has the advantage of performing both state and parameter estimation simultaneously, as shown in the examples of Chapter V.

Consider the class of systems defined by

$$\begin{aligned} \dot{\underline{x}} &= \underline{g}(t,\underline{x}) + k(t,\underline{x})\underline{w} \\ \underline{z} &= \underline{h}(t,\underline{x}) + \underline{v} \end{aligned} \tag{2.6}$$

where x is the "generalized" state vector of dimension n and includes the unknown parameter vector which has been adjoined to the original state vector. Also,

$\underline{g}(t,\underline{x})$ = n vector function

$k(t,x)$ = n x p vector function

$\underline{w}$ = p vector unknown input

$\underline{h}(t,\underline{x})$ = m vector function

$\underline{z}$ = m vector output

$\underline{v}$ = m vector measurement error.

It is desired to find the least square estimate of $\underline{x}$, designated $\hat{\underline{x}}$, which minimizes the cost function

$$J' = \int_0^T [\,\|\underline{z} - \underline{h}(t,\underline{x})\|^2_{W1} + \|\dot{\underline{x}} - \underline{g}(t,\underline{x})\|^2_{W2}]\, dt \tag{2.7}$$

or, alternately written as

$$J' = \int_0^T [\| \underline{v} \|^2_{W1} + \| \underline{w} \|^2_{k'W2k}] \, dt \qquad (2.8)$$

where W1 and W2 are weighting matrices which determine the relative weighting to be placed on the individual terms in the cost function. The method of least squares is used primarily due to historical precedent. Since its discovery by Gauss [12], it has been used with much success on many estimation problems.

By writing the Hamiltonian and making use of the maximum principle, a two-point boundary-value problem results for which some of the boundary conditions are specified at t = 0 and some at t = T, where now the variable T is regarded as the independent variable. By further utilizing the invariant imbedding equation, which is derived in Appendix A, the following set of sequential estimation equations are obtained.

$$\dot{\hat{\underline{x}}} = \underline{g}(T,\hat{\underline{x}}) + 2P(T)\, H\, (T,\hat{\underline{x}})\, W1\, [\underline{z}(T) - \underline{h}(T,\hat{\underline{x}})]$$

$$\begin{aligned} \dot{P} = {} & g_{\hat{\underline{x}}}(T,\hat{\underline{x}})\, P + P g'_{\hat{\underline{x}}}\, (T,\hat{\underline{x}}) \\ & + 2P[H\, W1\, \{\underline{z}(t) - \underline{h}(T,\hat{\underline{x}})\}]_{\hat{\underline{x}}}\, P \\ & + \frac{1}{2}\, \underline{k}(T,\hat{\underline{x}})\, V^{-1}\, (T,\hat{\underline{x}}) k'\, (T,\hat{\underline{x}}) \end{aligned} \qquad (2.9)$$

where

$$g_{\hat{\underline{x}}} = \frac{\partial \underline{g}}{\partial \hat{\underline{x}}}$$

$$H = \left[\frac{\partial \underline{h}}{\partial \hat{\underline{x}}}\right]'$$

$$V(T,\hat{\underline{x}}) = k'(T,\hat{\underline{x}})\, W2\, k(T,\hat{\underline{x}})$$

and $P(T) = n \times n$ matrix.

The sequential nature of these estimation equations is brought out by the fact that T is a running variable. The derivation of these equations is given in detail in Appendix B.

Summary

This chapter has presented various linear time-invariant models which can be used to approximate nonlinear time-varying systems. A method has been introduced which will identify these models in a stochastic environment in an on-line fashion. These results will be combined with the methods of control presented in Chapters III and IV to provide a suboptimal adaptive system.

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CHAPTER III

SUBOPTIMAL ADAPTIVE REGULATOR CONTROL

When a system undergoes unforeseen changes or disturbances, it is impossible to derive a precalculated control which will force the system to operate in some optimal fashion for all time. It then becomes necessary to monitor the system, and as the system deviations are noted, the control law can be altered in such a way as to maintain an acceptable system performance. Thus the overall system can be regarded as adaptive. Although there is no definition at present for an adaptive system which meets with general acceptance, Kalman [1] gives the following definition:

> A control system is adaptive if it is possible for it to change its control law as a result of measured changes of the control object and its environment and in such a way as to operate at all times in an optimal or nearly optimal fashion.

The successful operation of an adaptive control system will depend on the estimation of the state variables and the identification of the system dynamics. This is the problem considered in Chapter II. In this chapter it is desired to introduce a feasible method for the calculation of an effective control law. The derivation of this method results from an emphasis being placed on the reduction of complex calculations, thereby reducing the calculation time. This hopefully allows the controller to function in an on-line fashion.

Derivation of Control Law

Assume a given nonlinear system of the form

$$\underline{\dot{x}} = f\ (\underline{x},\ \underline{u},\ t)$$
$$\underline{x}\ (t_o) = \underline{x}_o, \tag{3.1}$$

where $\underline{x}$ is an n-dimensional vector and $\underline{u}$ is an r-dimensional vector. It is desired to control this system over the time interval $t \epsilon (t_o, t_f)$ while minimizing the performance index

$$J = \theta[\underline{x}(t_f)] + \int_{t_o}^{t_f} \Phi(\underline{x},\underline{u},t)\ dt. \tag{3.2}$$

The nonlinear system is identified as a linear system

$$\underline{\dot{x}} = A(t_i)\ \underline{x}\ (t) + B(t_i)\ \underline{u}\ (t) \tag{3.3}$$

and the cost function approximated as

$$J = \frac{1}{2}\ \underline{\tilde{x}}'(t_f)W\ \underline{\tilde{x}}\ (t_f) + \frac{1}{2} \int_{t_o}^{t_f} [\underline{\tilde{x}}'Q\ \underline{\tilde{x}} + \underline{u}'R\ \underline{u}]\ dt \tag{3.4}$$

where

$$\underline{\tilde{x}} = \underline{x}(t) - \underline{x}_d \tag{3.5}$$

and where $\underline{x}_d$ represents the desired final state of the system at time t_f. The sequential suboptimal control is obtained by first dividing the time interval of control into N subintervals $(t_{i+1} - t_i)$ where $t_i \epsilon (t_o, t_f)$ and $i = 0,1,\ldots,\ N - 1$. The nonlinear system (3.1) is identified as a linear system (3.3) over the interval $t \epsilon (t_i, t_f)$.

An optimum control is then found which will minimize, not the given cost function (3.2), but a related cost function.

$$J = \frac{1}{2}\,\underline{\tilde{x}}'(t_f)W\,\underline{\tilde{x}}\,(t_f) + \frac{1}{2}\int_{t_i}^{t_f} C(t)[\underline{\tilde{x}}'Q\,\underline{\tilde{x}} + \underline{u}'\,R\underline{u}]\,dt\,. \qquad (3.6)$$

Once the control is found, it is applied over the time interval $t\epsilon(t_i,t_{i+1})$. It is significant to note that even though the A and B matrices are identified at time t_i, based on information up to time t_i, A and B are assumed to remain constant throughout $t\epsilon(t_i,t_f)$. This is not unreasonable since C(t), a weighting term, may be selected such as to offset any error introduced by the assumption.

The choice of C(t) is thus very important. Although (3.3) is linear time-invariant, the inclusion of C(t) in (3.6) will, in general, lead to a two-point boundary-value problem for which the canonic equations are time varying. These can be solved, but the solution time is often incompatible with the requirement for on-line control. To circumvent this difficulty, it is advantageous to let

$$C(t) = e^{\omega t} \qquad (3.7)$$

where ω is some constant to be determined by the particular system to be controlled. Using (3.7) the resulting two-point boundary-value problem can be easily solved.

To show this, take the system

$$\begin{aligned}\dot{\underline{x}} &= A\underline{x} + B\underline{u}\\ \underline{x}(t_i) &= \underline{x}_i,\end{aligned} \qquad (3.8)$$

and the performance index

$$V = \frac{1}{2}\int_{t_i}^{t_f} [\underline{x}'Q\,\underline{x} + \underline{u}'R\,\underline{u}]\, e^{\omega t}\, dt. \tag{3.9}$$

The maximum principle of Pontryagin [2] is well suited for this type of problem and is used here. Its application is well known and may be found in many references [3,4,5,6]. Therefore its proof is not given here and only the results are used.

For the system given by (3.8) and the performance index given by (3.9) the Hamiltonian is given by

$$H = \frac{1}{2}[\underline{x}'Q\,\underline{x} + \underline{u}'R\,\underline{u}]\, e^{\omega t} + \underline{\lambda}'[A\underline{x} + B\underline{u}] \tag{3.10}$$

where $\underline{\lambda}$ is the n-dimensional Lagrange multiplier vector. The optimal control is found by equating

$$\frac{\partial H}{\partial \underline{u}} = \underline{0} \tag{3.11}$$

to give

$$\underline{u} = -R^{-1}\, B'\underline{\lambda}\, e^{-\omega t}. \tag{3.12}$$

The canonic equations are given by

$$\dot{\underline{x}} = \frac{\partial H}{\partial \underline{\lambda}} \tag{3.13}$$

$$\dot{\underline{\lambda}} = -\frac{\partial H}{\partial \underline{x}} \tag{3.14}$$

so that

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \tag{3.15}$$

$$\dot{\underline{\lambda}} = -Q\, e^{\omega t}\, \underline{x} - A'\underline{\lambda}\ . \tag{3.16}$$

The endpoint conditions on $\underline{\lambda}$ are given by the transversality condition

$$\underline{\lambda}\ (t_f) = \underline{0}. \tag{3.17}$$

Create a $\underline{\sigma}$ vector of degree n-r and define it to be

$$\underline{\sigma} = e^{-\omega t} \begin{bmatrix} \lambda_{r+1} \\ - \\ - \\ - \\ \lambda_n \end{bmatrix} . \tag{3.18}$$

Adjoin this $\underline{\sigma}$ vector to the $\underline{u}$ vector yielding a n-dimensional vector. This new vector can be written as

$$\left[\begin{array}{c} \underline{u} \\ \hline \underline{\sigma} \end{array}\right] = \left[\begin{array}{c|c} -R^{-1} & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c|c} \multicolumn{2}{c}{B'} \\ \hline 0 & I \end{array}\right] \underline{\lambda}\ \ e^{-\omega t} : \tag{3.19}$$

Let

$$M = \left[\begin{array}{c|c} -R^{-1} & 0 \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c|c} \multicolumn{2}{c}{B'} \\ \hline 0 & I \end{array}\right] . \tag{3.20}$$

Now,

$$\underline{\lambda} \quad = M^{-1} \left[\begin{array}{c} \underline{u} \\ \hline \underline{\sigma} \end{array}\right] e^{\omega t} \tag{3.21}$$

and

$$\underline{\dot{\lambda}} \quad = M^{-1} \left[\begin{array}{c} \underline{\dot{u}} \\ \hline \underline{\dot{\sigma}} \end{array}\right] e^{\omega t} + \omega M^{-1} \left[\begin{array}{c} \underline{u} \\ \hline \underline{\sigma} \end{array}\right] e^{\omega t}. \tag{3.22}$$

Using (3.21) and (3.22), (3.16) simplifies to

$$\dot{\underline{\Gamma}} = \Omega\, \underline{x} + \Theta\, \underline{\Gamma} \tag{3.23}$$

where

$$\underline{\Gamma} = \begin{bmatrix} \underline{u} \\ \hline \underline{\sigma} \end{bmatrix} \tag{3.24}$$

$$\Omega = -\, M\, Q$$

$$\Theta = -\, [M\, A'\, M^{-1} - \omega I].$$

By further writing

$$B\underline{u} = [B \mid 0] \begin{bmatrix} \underline{u} \\ \hline \underline{\sigma} \end{bmatrix} = C \begin{bmatrix} \underline{u} \\ \hline \underline{\sigma} \end{bmatrix} = C\, \underline{\Gamma} \tag{3.25}$$

equations (3.15) and (3.16) become

$$\dot{\underline{x}} = A\underline{x} + C\, \underline{\Gamma} \tag{3.26}$$

$$\dot{\underline{\Gamma}} = \Omega\underline{x} + \Theta\, \underline{\Gamma} \tag{3.27}$$

with

$$\underline{x}(t_i) = \underline{x}_i \tag{3.28}$$

and

$$\underline{\Gamma}(t_f) = \underline{0}\ . \tag{3.29}$$

Having obtained this linear time-invariant set of equations, the solution may be written in the form

$$\begin{bmatrix} \underline{x}\,(t) \\ \hline \underline{\Gamma}\,(t) \end{bmatrix} = e^{\left[\begin{array}{c|c} A & C \\ \hline \Omega & \Phi \end{array}\right](t\,-\,t_i)} \begin{bmatrix} \underline{x}\,(t_i) \\ \hline \underline{\Gamma}\,(t_i) \end{bmatrix}$$

$$\begin{bmatrix} \underline{x}\,(t) \\ \hline \underline{\Gamma}\,(t) \end{bmatrix} = e^{S(t_i)(t-t_i)} \begin{bmatrix} \underline{x}\,(t_i) \\ \hline \underline{\Gamma}\,(t_i) \end{bmatrix} \tag{3.30}$$

or,

$$\begin{bmatrix} \underline{x}\,(t) \\ \hline \underline{\Gamma}\,(t) \end{bmatrix} = \Phi(t-t_i) \begin{bmatrix} \underline{x}\,(t_i) \\ \hline \underline{\Gamma}\,(t_i) \end{bmatrix}$$

$$\begin{bmatrix} \underline{x}\,(t) \\ \hline \underline{\Gamma}\,(t) \end{bmatrix} = \left[\begin{array}{c|c} \Phi_{xx}(t-t_i) & \Phi_{x\Gamma}(t-t_i) \\ \hline \Phi_{\Gamma x}(t-t_i) & \Phi_{\Gamma\Gamma}(t-t_i) \end{array}\right] \begin{bmatrix} \underline{x}\,(t_i) \\ \hline \underline{\Gamma}\,(t_i) \end{bmatrix} . \tag{3.31}$$

$\underline{\Gamma}(t_i)$ is unknown, but can be easily found from

$$\underline{\Gamma}(t) = \Phi_{\Gamma x}(t-t_i)\ \underline{x}\ (t_i) + \Phi_{\Gamma\Gamma}(t-t_i)\ \underline{\Gamma}(t_i). \tag{3.32}$$

Since

$$\underline{\Gamma}(t_f) = \underline{0},$$

$$\underline{\Gamma}(t_i) = -\ \Phi_{\Gamma x}^{-1}(T_i)\ \Phi_{\Gamma\Gamma}(T_i)\ \underline{x}\ (t_i) \tag{3.33}$$

where $T_i = t_f - t_i$ is the time to go. Now, the first r components, which comprise the control vector, can be calculated using (3.33).

For sufficiently small subintervals, $t\varepsilon(t_i, t_{i+1})$, it is feasible to let the control remain constant. Thus, it is necessary to calculate only $\underline{u}(t_i)$. This sometimes necessitates using a smaller subinterval than would otherwise be required. If this does give unsatisfactory results, then there should still be no problem in operating on-line since the control can be computed in real time using (3.32) and applied as it is computed.

Although the derivation shown here assumes the performance index given in (3.9), a slight reformulation yields similar results for the performance index given in (3.6).

Matrix Exponential

The most prohibitive factor in using this scheme in an on-line fashion is the calculation of the matrix exponential exp $\{S(t_i)(t_f-t_i)\}$ to find the transition matrix $\Phi(t_f-t_i)$. This matrix exponential and some of its properties are discussed by Kalman [1], along with some of the accuracy limitations encountered in its calculation. Since it is calculated by taking a finite number of terms in a Taylor series, calculation times become important if too many terms are needed before the desired accuracy is reached. This of course can occur when the elements of the matrix $S(t_i)(t_f-t_i)$ are large. One way to circumvent this difficulty is to let

$$e^{S(t_i)(t_f-t_i)} = e^{\sum_{j=1}^{m} \frac{1}{m} S(t_i)(t_f-t_i)}$$

or

$$e^{S(t_i)(t_f-t_i)} = \left[e^{\frac{1}{m} S(t_i)(t_f-t_i)}\right]^m . \qquad (3.34)$$

Now the elements of the matrix used in the Taylor series have been reduced by a factor of m, thus allowing much faster convergence to the correct answer. This results in the m^{th} root of the transition matrix, and is therefore multiplied by itself (m-1) times to yield the transition matrix. In general, considerable difference results in calculation times even for $m = 2$.

If large subintervals are used such that the control must be recalculated every Δt seconds during the subinterval $t\epsilon(t_i, t_{i+1})$, then the following property of the transition matrix, valid for time invariant systems, may be used.

$$\Phi[n(t-t_o)] = [\Phi(t-t_o)]^n . \qquad (3.35)$$

This reduces the problem of calculating a matrix exponential every Δt seconds to a simple matrix multiplication.

Summary

This chapter has attempted to present a suboptimal adaptive regulator control scheme for a nonlinear system with a given cost function. The nonlinear system is modeled by a linear time-carying system of assumed form. This model is assumed stationary over subintervals of time which allows a controller to generate a sequential control law

in an on-line fashion which minimizes an integral of time weighted quadratic form of error and control effort.

The emphasis has been placed on computational simplicity and a discussion has been presented on the matrix exponential and its calculation.

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CHAPTER IV

SUBOPTIMAL ADAPTIVE TRAJECTORY CONTROL

In some instances it may be worthwhile to force a system to follow either a predetermined desirable trajectory or possibly a precalculated trajectory which is optimal in some sense. Trajectory tracking methods are particularly suited to systems which are difficult to control or perhaps need to be constrained within certain limits. However, this form of control may result in a somewhat higher overall cost since it does not allow the system to adapt to a new trajectory when environmental changes or noise are present.

Several schemes have been developed for accomplishing trajectory control. For example, the control of a typical lifting reentry vehicle about a nominal trajectory has been investigated by Kovatch [1] using linearized equations about the nominal trajectory and a quadratic cost function. This particular method requires the storage of the precomputed nominal trajectory and a set of precomputed feedback gains. The controller acts on deviations of the state variables from this nominal trajectory by using the precalculated gains. This is essentially a form of open loop control and is not desirable when the system may be unknown a priori. Also, large memory requirements are needed which is typical for most trajectory tracking schemes.

Breakwell, Speyer, and Bryson [2] propose a method of control which minimizes a terminal quantity while satisfying specified terminal conditions in the presence of small disturbances. The scheme is based on a linear perturbation from a nominal optimum path and involves the use of the second variation of the calculus of variations.

The method presented in this chapter insures acceptable performance if the system model is well defined and enough points are taken along the desired trajectory. The overall system is adaptive since the system model is identified in a sequential manner, and an appropriate control is then determined which forces the system to operate on or near the desired trajectory. It should be noted that the emphasis is placed on the simplicity of the control, the ease, and speed with which it may be computed, and possible minimization of memory requirements.

Derivation of Control Law

Assume a system, such as (3.1), which is to track a given trajectory over the interval $t \epsilon (t_o, t_f)$. The given trajectory may be one which is dictated by certain problem constraints or simply desired responses. In other instances, it may be the result of the minimization of a performance index as in (3.2). In either case the state vector for the desired trajectory for $t = t_1, t_2, \ldots, t_m$ is stored for future use and designated $\underline{x}_d(t_i)$. If the time increments $\Delta t_i = t_{i+1} - t_i$ are equal, $m\Delta t_o = t_f - t_o = m\Delta t_i$.

For the suboptimal control scheme the system is identified as

$$\underline{\dot{x}} = A(t_i)\ \underline{x} + B(t_i)\ \underline{u} \tag{4.1}$$

where A and B are constant matrices over the interval $t\epsilon(t_i, t_{i+1})$. For each subinterval, a cost function of the form

$$V_i = \frac{1}{2}[\underline{x}(t_{i+1}) - \underline{x}_d(t_{i+1})]'\ P[\underline{x}(t_{i+1}) - x_d(t_{i+1})] + \frac{1}{2}\int_{t_i}^{t_{i+1}} \underline{u}'\ R(t_i)\ \underline{u}\ dt \tag{4.2}$$

is chosen. The elements of P determine how closely the predetermined trajectory should be followed. The total cost for the time interval $t\epsilon(t_o, t_f)$ is then

$$\varphi = \sum_{i=0}^{n-1} V_i\ . \tag{4.3}$$

At each $t = t_i$ the two-point boundary-value problem must be solved, at least for the initial values of the control. As mentioned in Chapter III, if the subinterval length, $t\epsilon(t_i, t_{i+1})$, is small, the control can be held constant over the entire subinterval with essentially the same result as if it is varied.

The suboptimal control can be found using the maximum principle of Pontryagin [3]. The Hamiltonian can be written as

$$H = \frac{1}{2}\underline{u}'\ R\underline{u} + \underline{\lambda}'[A(t_i)\ \underline{x} + B(t_i)\ \underline{u}]. \tag{4.4}$$

For simplicity, it is desirable to write $A(t_i)$ and $B(t_i)$ as A and B since they are constant matrices over each subinterval. The canonic equations yield,

$$\underline{u} = -R^{-1} B' \underline{\lambda} \tag{4.5}$$

$$\dot{\underline{x}} = A \underline{x} + B \underline{u} \tag{4.6}$$

$$\dot{\underline{\lambda}} = -A' \underline{\lambda} \tag{4.7}$$

$$\underline{x}(t_i) = \underline{x}_i \tag{4.8}$$

and the transversality condition at $t = t_{i+1}$ gives

$$\underline{\lambda}(t_{i+1}) = P[\underline{x}(t_{i+1}) - \underline{x}_d(t_{i+1})]. \tag{4.9}$$

It appears desirable to convert the equations (4.5) through (4.9) into equations for which the control can be computed directly. With this in mind, define an n-r dimension vector $\underline{\sigma}$ as being

$$\underline{\sigma} = \begin{bmatrix} \lambda_{r+1} \\ \vdots \\ \lambda_n \end{bmatrix} . \tag{4.10}$$

Adjoin this $\underline{\sigma}$ vector to the $\underline{u}$ vector so that

$$\begin{bmatrix} \underline{u} \\ \hline \underline{\sigma} \end{bmatrix} = \begin{bmatrix} -R^{-1} B' \\ \hline 0 \;\vdots\; I \end{bmatrix} [\underline{\lambda}] \tag{4.11}$$

where I is an r x r identity matrix. Let

$$M = \begin{bmatrix} -R^{-1} B' \\ \hline 0 \;\vdots\; I \end{bmatrix} \tag{4.12}$$

and

$$\underline{\Gamma} = \begin{bmatrix} \underline{u} \\ \hline \underline{\sigma} \end{bmatrix} \tag{4.13}$$

Thus,

$$\underline{\lambda} = M^{-1}\,\underline{\Gamma} \tag{4.14}$$

and

$$\dot{\underline{\lambda}} = M^{-1}\,\dot{\underline{\Gamma}}\ . \tag{4.15}$$

Then, equations (4.6) and (4.7) become

$$\dot{\underline{x}} = Ax + C\,\underline{\Gamma} \tag{4.16}$$

$$\dot{\underline{\Gamma}} = \Theta\,\underline{\Gamma} \tag{4.17}$$

where

$$\Theta = -M\,A'\,M^{-1} \tag{4.18}$$

and C is an n x n matrix defined by

$$C = [B \,|\, 0]. \tag{4.19}$$

The initial conditions on $\underline{x}$ are given by (4.8), and the endpoint conditions on $\underline{\Gamma}$ are given by

$$\underline{\Gamma}(t_{i+1}) = \begin{bmatrix} \underline{u}(t_{i+1}) \\ \hline \underline{\sigma}(t_{i+1}) \end{bmatrix}$$

$$\underline{\Gamma}(t_{i+1}) = \begin{bmatrix} -R^{-1}\ B' \\ \hline 0 \ \vdots\ I \end{bmatrix} P\,[\underline{x}(t_{i+1}) - \underline{x}_d(t_{i+1})]$$

$$\underline{\Gamma}(t_{i+1}) = M\,P[\underline{x}(t_{i+1}) - \underline{x}_d(t_{i+1})] \tag{4.20}$$

where I is an r x r identity matrix.

The solution of (4.16), (4.17), (4.20) and (4.8) is given by

$$\begin{bmatrix} \underline{x}(t-t_i) \\ \hline \underline{\Gamma}(t-t_i) \end{bmatrix} = e^{\left[\begin{array}{c|c} A & C \\ \hline 0 & \Theta \end{array}\right](t-t_i)} \begin{bmatrix} \underline{x}(t_i) \\ \hline \underline{\Gamma}(t_i) \end{bmatrix} \tag{4.21}$$

where $\underline{\Gamma}(t_i)$ is yet to be found. To do this, define

$$e^{\left[\begin{array}{c|c} A & C \\ \hline 0 & \Theta \end{array}\right](t-t_i)} = \left[\begin{array}{c|c} \Phi_{xx}(t-t_i) & \Phi_{x\Gamma}(t-t_i) \\ \hline \Phi_{\Gamma x}(t-t_i) & \Phi_{\Gamma\Gamma}(t-t_i) \end{array}\right] . \tag{4.22}$$

At $t = t_{i+1}$, $\underline{\Gamma}$ is known in terms of $\underline{x}$. Utilizing this information, $\underline{\Gamma}(t_i)$ is found to be

$$\underline{\Gamma}(t_i) = \gamma^{-1}[\psi\ \underline{x}(t_{i+1}) + P\ \underline{x}_d\ (t_{i+1})] \tag{4.23}$$

where

$$\gamma = P\ \Phi_{xr}(t_{i+1}-t_i) - \Phi_{\Gamma\Gamma}(t_{i+1}-t_i) \tag{4.24}$$

and

$$\psi = \Phi_{\Gamma x}(t_{i+1}-t_i) - P\ \Phi_{xx}(t_{i+1}-t_i). \tag{4.25}$$

Since the first r components of $\underline{\Gamma}$ comprise the control vector, the control can be applied after computing (4.23).

It is important to note that the given trajectory may be followed more closely by taking more points. However, this has the disadvantages of requiring more computation time and also larger memory storage. These may be critical factors and thus require careful attention. The situation can be helped possibly by choosing more points where the system response is changing rapidly and less points where the system response is changing slowly.

Matrix Inverse

The most time consuming calculations in this scheme are the calculation of the matrix exponential given in (4.22), and the calculation of the matrix inverse given in (4.23). Suggestions concerning the calculation of the matrix exponential are given in Chapter III. Although many methods have been devised for the calculation of the matrix inverse [4], there is no universal best method. It can be shown that a straightforward calculation of a matrix inverse from its definition,

$$\gamma^{-1} = \frac{\text{cofactor matrix of } \gamma}{\text{determinant of } \gamma} , \tag{4.26}$$

requires $(n!)(n^2-n-1)$ multiplications and n^2 divisions when the matrix γ is of dimension n [5]. For large n, this requires relatively large computer times. Also, since the elements of γ are

the result of physical measurements, each one will contain errors. In addition, limitations on word length, or numerical accuracy, of computational methods or the computer itself introduces some error for each multiplication or division. Thus, numerical accuracy is a direct function of the number of multiplications and divisions employed in the inversion algorithm used.

Occasionally, it may happen that the matrix to be inverted is singular, and the inverse does not exist. However, the concept of a "generalized inverse" [6] can be introduced to sidestep both the theoretical and practical problems of singular matrices.

Due to the obvious difficulties which may be encountered when obtaining a matrix inverse, care should be taken in the choice of a suitable computational algorithm. For the examples given in Chapter V, an algorithm which employs the Gaussian elimination method [7] is used and found very efficient with respect to both speed and accuracy.

The Gaussian elimination method indirectly gives the matrix inverse by first solving the equation $A\underline{x} = \underline{y}$, where A is an n x n matrix, $\underline{x}$ is an n vector, and $\underline{y}$ is an n vector. This equation is solved by the elimination of one unknown at a time through substitution. Note, however, that if $\underline{y}_i$ is defined as the vector whose components are all zero except for the i^{th} component which is equal to one, then the solution to $A\underline{x}_i = \underline{y}_i$ gives the i^{th} column of A^{-1}. Therefore, A^{-1} is found by solving the equation n times using y_1, y_2, ... , and y_n.

Summary

This chapter has attempted to present a method for the on-line suboptimal adaptive trajectory control of a nonlinear system. By the sequential identification of a linear constant-coefficient model, the given system is forced to track a predetermined trajectory at certain specified points. Therefore, an acceptable response can be obtained even though the complete system dynamics are not known a priori.

The actual system response can be forced to fit the desired given trajectory more closely, simply by specifying more points along the desired trajectory. However, this results in larger computation times and a need for more storage requirements.

A short discussion has been presented on the calculation of the matrix inverse and the difficulties encountered in its calculation.

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CHAPTER V

APPLICATIONS

The purpose of this chapter is to illustrate the effectiveness of the methods of control presented in Chapters III and IV. The three examples given are of practical importance and of current interest. They include the startup of a nuclear reactor, the orbital transfer of a vehicle using a low thrust ion engine, and the startup of a nuclear powered rocket engine. Each of the problems is nonlinear, and the different methods for choosing a linear model, as outlined in Chapter II, are used. The sequential scheme used for parameter and state estimation, which is also mentioned in Chapter II, proves to be very effective for these applications.

In each case, the results are compared with predetermined "optimum" or desired results. However, due to environmental changes or noise the adaptive controllers may substantially reduce the cost over that which results from a predetermined control. Although input and output noise is injected in two of the examples, no environmental changes are introduced so that the effectiveness of the methods can be illustrated by the comparisons with predetermined results.

All calculations were made on an IBM-709 digital computer at the University of Florida Computing Center, with the computer programs being written in Fortran IV.

Example 1: Nuclear Reactor Startup

The nuclear reactor kinetics equations [1] are given by

$$\dot{n} = \frac{\rho - \beta}{\Lambda} n + \lambda c \tag{5.1}$$

$$\dot{c} = \frac{\beta}{\Lambda} n - \lambda c \tag{5.2}$$

where

n = neutron flux density

c = precursor density

ρ = reactivity

β = 0.0064 = fractions of precursors formed

Λ = 0.001 seconds = neutron lifetime

λ = 0.1 seconds^{-1} = precursor decay constant.

It is desired to find the control which will drive the neutron flux density to a desired value while minimizing the integral of control, or reactivity, squared. That is, given (5.1) and (5.2) with

$$\begin{aligned} n(0) &= 0.5 \text{ k.w.} \\ c(0) &= 32.0 \text{ k.w.} \end{aligned} \tag{5.3}$$

find the reactivity, ρ, such that

$$n(1) = 5.0 \tag{5.4}$$

and the performance index

$$J = \frac{1}{2}\int_0^1 \rho^2 \, dt \tag{5.5}$$

is minimized.

The problem is further complicated by the presence of an input noise, w, and an output noise, v. The input noise, w, is simulated by a sawtooth waveform of zero mean, having a maximum magnitude of 0.0008 and a period of 0.02 seconds, and the output noise, v, is a similar waveform with a maximum magnitude of 0.25. Due to this noise and a reasonable doubt as to the validity of the assumptions made in the derivation of (5.1) and (5.2), it is worthwhile to attempt an on-line adaptive control of the reactor using the methods of Chapters III and IV.

The overall system is shown in Figure 2, where the reactor dynamics are given by

$$\dot{n} = -6.4\, n + 0.1\, c + 10^3\, n(\rho + w) \tag{5.6}$$

$$\dot{c} = 6.4\, n - 0.1\, c \tag{5.7}$$

$$z = n + v\ . \tag{5.8}$$

Note the presence of the input noise, w, and the output measurement noise, v. The model used for control at each $t = t_i$ is given by

$$\dot{n} = -6.4\, n + 10^3\, n(t_i)\rho + K(t_i) \tag{5.9}$$

where $n(t_i)$ is the estimate of n at $t = t_i$, or $\hat{n}(t_i)$, and $K(t_i)$ is an unknown parameter to be identified. Note that the second-order

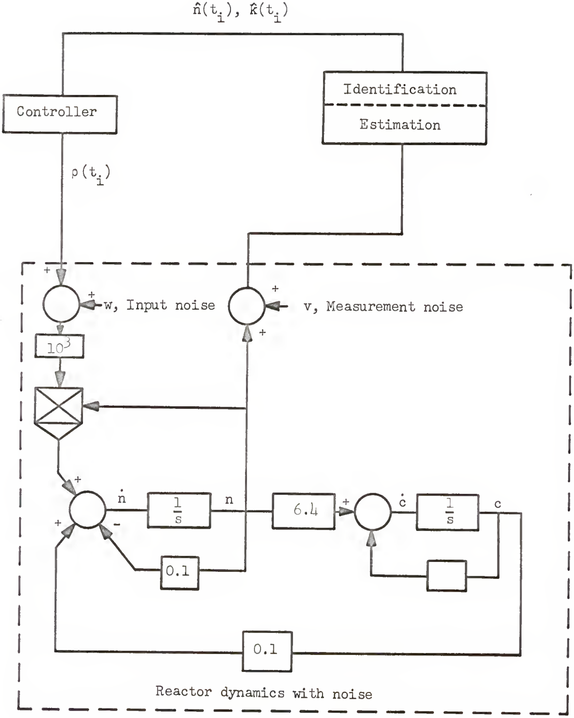

Figure 2. Adaptive control of nuclear reactor

nonlinear system is reduced to a first-order linear system. It is, of course, desired to find $\rho(t_i)$.

The model used for the estimation of $n(t_i)$ and $K(t_i)$, used in (5.9), is given by

$$\dot{\hat{n}} = -6.4\,\hat{n} + 10^3\,\hat{n}(\rho + w) + \hat{K} \tag{5.10}$$

$$\dot{\hat{K}} = 0 \tag{5.11}$$

$$z = \hat{n} + v \tag{5.12}$$

where it is necessary to estimate n and identify K, and ρ is known for all past time, $t < t_i$. Using equations (2.6), (2.8), and (2.9), the following sequential estimation equations are derived.

$$\begin{aligned}
\dot{\hat{n}} &= [-6.4 + 10^3\,\rho]n + \hat{K} + 2W1(z-n)\,P_{11} \\
\dot{\hat{K}} &= 2W1(z-n)\,P_{21} \\
\dot{P}_{11} &= 2[-6.4 + 10^3\rho]P_{11} + P_{12} + P_{12} - 2W1\,P_{11}^2 + \frac{1}{2W2} \\
\dot{P}_{12} &= [-6.4 + 10^3\,\rho]P_{12} + P_{22} - 2W1\,P_{11}\,P_{12} \\
\dot{P}_{21} &= [-6.4 + 10^3\,\rho]P_{21} + P_{22} - 2W1\,P_{11}\,P_{21} \\
\dot{P}_{22} &= -2W1\,P_{12}\,P_{21}\,.
\end{aligned} \tag{5.13}$$

Note that $P_{12} = P_{21}$, and because the estimation problem is linear, neither $\dot{n}$ nor $\dot{K}$ appears in the $\dot{P}$ equations.

Regulator Control

For the regulator type of control presented in Chapter III a cost function of the form

$$J = \frac{1}{2} R[n(1) - 5.0]^2 + \frac{1}{2} \int_{t_i}^{1.0} \rho^2 e^{\omega t} dt \tag{5.14}$$

is used. By using equation (5.9), the Hamiltonian is written as

$$H = \frac{1}{2} e^{\omega t} \rho^2 + \lambda_1[-6.4 n + 10^3 n(t_i)\rho] + K(t_i). \tag{5.15}$$

The canonic equations, (3.13) and (3.14), along with (3.18) through (3.25), yield

$$\begin{aligned} \dot{n} &= -6.4 n + 10^3 n(t_i)\rho + K(t_i) \\ \dot{K} &= 0 \\ \dot{\rho} &= -(\omega - 6.4)\rho \\ \dot{\sigma} &= \frac{\rho}{10^3 n(t_i)} - \omega \sigma \end{aligned} \tag{5.16}$$

with the transversality condition

$$\rho(1) = -10^3 n(t_i) R e^{-\omega(1.0-t_i)} [n(1) - 5.0], \tag{5.17}$$

and $\sigma(1) = 0.0$.

By the proper manipulation of (5.16), and the substitution of (5.17), an analytical expression for $\rho(t_i)$ is found.

$$\rho(t_i) = \frac{\left[n(t_i) - \frac{K(t_i)}{6.4}\right] e^{-6.4(1.0-t_i)} + \frac{K(t_i)}{6.4} - 5.0}{-\frac{10^3 n(t_i)}{\omega-12.8}\left[e^{-6.4(1.0-t_i)} - e^{-(\omega-6.4)(1.0-t_i)}\right] - \frac{1.0\, e^{+6.4(1.0-t_i)}}{R\, 10^3 n(t_i)}} . \tag{5.18}$$

Thus, due to the low order of the system model, the calculation of the matrix exponential can be waived, and the control can be found by substituting t_i, $n(t_i)$, and $K(t_i)$ into (5.18).

The results of this approach are given in Figures 3, 4 and 5, with $R = 0.001$, $\omega = 3.0$, and a subinterval size, $(t_{i+1}-t_i)$, of 0.05. The initial condition matrix for P is given by

$$P(0) = \begin{bmatrix} 20 & 5 \\ 5 & 20 \end{bmatrix}$$

with W1 = 10, and W2 = 10, $\hat{n}(0) = 0.5$, and $\hat{K}(0) = 3.5$.

Trajectory Control

For the trajectory type of adaptive control, a set of points are taken from the predetermined optimum trajectory for n, the neutron flux density. These values, $n_d(t_{i+1})$, are equally spaced in time so that $\Delta t = t_{i+1} - t_i = 0.05$ seconds. The cost function used is

$$V_i = \frac{1}{2} R[n(t_{i+1}) - n_d(t_{i+1})]^2 + \frac{1}{2}\int_{t_i}^{t_{i+1}} \rho^2 \, dt. \tag{5.19}$$

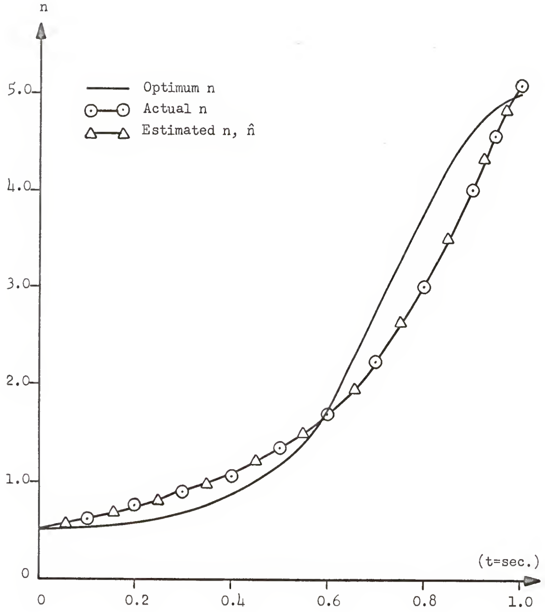

Figure 3. Nuclear reactor startup using regulator control

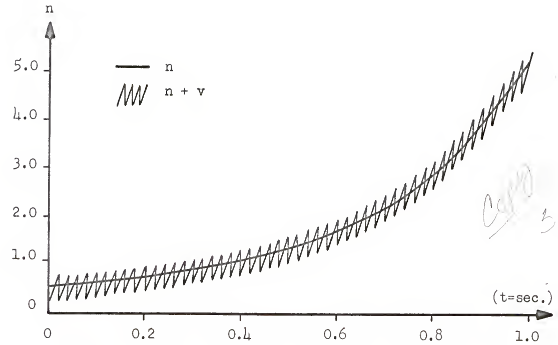

Figure 4. Neutron flux density showing effect of noise

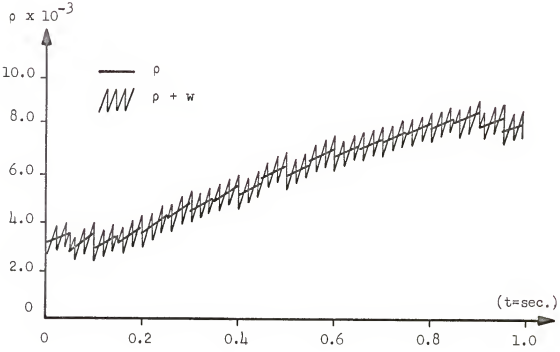

Figure 5. Reactivity showing effect of noise

Due to the similarity between (5.19) and (5.14) and since the system model is the same as for the regulator control, the result for $\rho(t_i)$ is similar to (5.18) with $n_d(1) = 5.0$ being replaced by $n_d(t_{i+1})$, $(1.0-t_i)$ being replaced by $\Delta t = t_{i+1} - t_i$, and ω being set equal to zero. Therefore,

$$\rho(t_i) = \frac{\left[n(t_i) - \frac{K(t_i)}{6.4}\right] e^{-6.4\,\Delta t} + \frac{K(t_i)}{6.4} - n_d(t_{i+1})}{\frac{10^3 n(t_i)}{12.8}\left[e^{-6.4\,\Delta t} - e^{6.4\,\Delta t}\right] - \left[\frac{1.0}{R\ 10^3 n(t_i)}\right] e^{6.4\,\Delta t}} . \tag{5.20}$$

The results are given in Figure 6 with $R = 0.1$ and with the initial condition matrix for P being

$$P(0) = \begin{bmatrix} 20 & 5 \\ 5 & 20 \end{bmatrix} .$$

Also, $W1 = 10$, $W2 = 10$, $\hat{n}(0) = 0.5$, and $\hat{K}(0) = 3.4$.

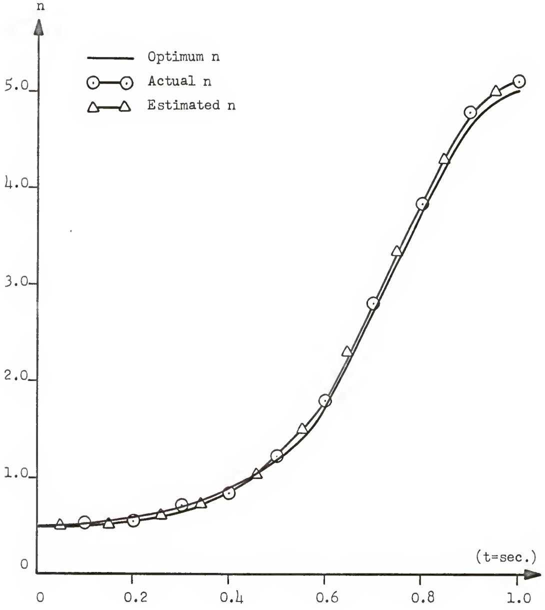

Figure 6. Nuclear reactor startup using trajectory control

Example 2: Low Thrust Orbital Transfer

As an example of trajectory control, consider the problem of minimizing the fuel consumption of a low thrust rocket which is to transfer from the orbit of Earth to the orbit of Mars in fixed time. The orbits of Mars and Earth are assumed to be circular and coplanar, and the gravitational attractions of the two planets are neglected. The problem has been previously formulated and solved for the open loop control assuming an inequality constraint on propellant mass flow, β, or thrust [2]. The normalized dynamics and boundary conditions are given by

$$\dot{r} = w \qquad \text{(Radial velocity)}$$

$$\dot{w} = \frac{v^2}{r} - \frac{K}{r^2} + \frac{C}{m}\sin\theta \qquad \text{(Radial acceleration)} \tag{5.21}$$

$$\dot{v} = -\frac{wv}{r} + \frac{C}{m}\cos\theta \qquad \text{(Circumferential acceleration)}$$

$$\dot{m} = -\beta \qquad \text{(Mass flow)}$$

$$\begin{array}{ll} r(0) = 1.0 & r(t_f) = 1.52 \\ w(0) = 0.0 & w(t_f) = 0.00 \\ v(0) = 1.0 & v(t_f) = 0.81 \\ m(0) = 1.0 & m(t_f) \quad \text{open} \end{array}$$

with $K = 1.00$

$C = 1.872$

$\beta_{max} = 0.075$

$\beta_{min} = 0.0$

where the final time, t_f, is 3.816 units which corresponds to 222.0 days and θ is the thrust angle measured from the local horizontal. It is desired to minimize the fuel consumed or equivalently the cost function

$$P = -m(t_f).$$

Computational results have been obtained using the iterative method of quasilinearization which show that the open-loop control is a bang-off bang type. However, in actual practice, due to measurement errors, noise, etc., it may not be desirable to apply this as a pre-calculated open-loop control, especially since near impact the system may require more than the open-loop β_{max} to match the critical end-points. For this reason, trajectory control seems feasible since it monitors the system and tends to keep it tracking the precalculated trajectory, although possibly at more cost.

Since $\dot{r}$ and $\dot{m}$ are relatively small, it seems feasible to model the system at each $t = t_i$ with a second order system of the form

$$\begin{aligned} \dot{w} &= \frac{v(t_i)}{r(t_i)} v - \frac{K}{r^2(t_i)} + \frac{C}{m(t_i)} u_i \\ \dot{v} &= \frac{v(t_i)}{r(t_i)} w + \frac{C}{m(t_i)} u_2 \end{aligned} \tag{5.22}$$

with

$$\begin{aligned} u_1 &= \beta \sin \theta \\ u_2 &= \beta \cos \theta. \end{aligned} \tag{5.23}$$

The total time for the flight (222 days) is divided into 37 subintervals $t\epsilon(t_i, t_{i+1})$ where $i = 0, 1, \ldots, 36$. Thus each subinterval corresponds to six days. It is desired to minimize the performance index, given by

$$V_i = \frac{1}{2} R_{11}[w(t_{i+1}) - w_d(t_{i+1})]^2 + \frac{1}{2} R_{22}[v(t_{i+1}) - v_d(t_{i+1})]^2 + \frac{1}{2}\int_{t_i}^{t_{i+1}} \alpha(u_1^2 + u_2^2)\, dt \qquad (5.24)$$

with $R_{11} = 1000$, $R_{22} = 1000$, and $\alpha = 1$. The values for w_d and v_d are taken from the optimal open-loop trajectory.

Using the maximum principle, the resulting canonic equations are

$$\begin{aligned}
\dot{w} &= \frac{v(t_i)}{r(t_i)} v - \frac{K}{r^2(t_i)} + \frac{C}{m(t_i)} u_1 \\
\dot{v} &= -\frac{v(t_i)}{r(t_i)} w + \frac{C}{m(t_i)} u_2 \\
\dot{u}_1 &= \frac{v(t_i)}{r(t_i)} u_2 \\
\dot{u}_2 &= -\frac{v(t_i)}{r(t_i)} u_1 .
\end{aligned} \qquad (5.25)$$

Note that (5.31) can be put in the form of (4.16) and (4.17) by adjoining the equation

$$\dot{s} = 0 \qquad (5.26)$$

to (5.25) where $s = -\dfrac{k}{r^2(t_i)}$.

The transversality condition, (4.20), yields

$$u_1(t_{i+1}) = -\frac{R_{11}C}{m\alpha}[w(t_{i+1}) - w_d(t_{i+1})]$$
$$u_2(t_{i+1}) = -\frac{R_{22}C}{m\alpha}[v(t_{i+1}) - v_d(t_{i+1})]. \tag{5.27}$$

Using the matrix exponential form of solution, given by (4.21), (4.22), and (4.23), $u_1(t_i)$ and $u_2(t_i)$ can be found.

The presence of an input noise, I, and two components of output measurement noise, N1 and N2, which add to w and v, respectively, requires that estimates of r, w, v, and m be obtained for use in (5.25) and (5.27). I, N1, and $N2_3$ are all chosen to be sawtooth waveforms of zero mean, with maximum magnitudes of 0.005, 0.005, and 0.05, respectively, and periods of 5.5 days, 4.5 days, and 3.5 days, respectively. Since the system is a fourth-order system, the P matrix of the estimator is a 4 x 4 matrix. Thus, although the P matrix is symmetrical, the estimation scheme requires the simultaneous solution of fourteen differential equations. For this reason it seems feasible to use an estimation model of the form

$$\dot{\hat{w}} = \frac{\hat{v}^2}{\hat{r}(t-\delta t)} - \frac{K}{\hat{r}(t-\delta t)^2} + \frac{C}{\hat{m}(t-\delta t)}(\beta+I)\sin\theta$$
$$\dot{\hat{v}} = -\frac{\hat{w}\hat{v}}{\hat{r}(t-\delta t)} + \frac{C}{\hat{m}(t-\delta t)}(\beta+I)\cos\theta \tag{5.28}$$

$$z_1 = \hat{w} + N1$$
$$z_2 = \hat{v} + N2.$$

Note that since r and m are not estimated directly, they are estimated by integrating $\hat{w}$ and $-\beta$, respectively, and then used in (5.28). This introduces a delay factor of δt seconds for these terms in (5.28), however, this has little effect on the results.

Using the estimation procedure of Chapter II, (2.9) gives

$$\begin{aligned}
\dot{\hat{w}} &= \frac{\hat{v}^2}{\hat{r}(t-\delta t)} - \frac{K}{\hat{r}(t-\delta t)^2} + \frac{C \sin \theta}{\hat{m}(t-\delta t)} + 2P_{11}W1\,(z_1-\hat{w}) + 2P_{12}W1\,(z_2-\hat{v}) \\
\dot{\hat{v}} &= -\frac{\hat{w}\hat{v}}{\hat{r}(t-\delta t)} + \frac{C \cos \theta}{\hat{m}(t-\delta t)} + 2P_{21}W1\,(z_1-\hat{w}) + 2P_{22}W1\,(z_2-\hat{v}) \\
\dot{P}_{11} &= \frac{2\hat{v}}{\hat{r}(t-\delta t)}(P_{21}+P_{12}) - 2W1\,(P_{11}^2+P_{12}P_{21}) + \frac{1}{2}\frac{\sin^2\theta}{W2} \\
\dot{P}_{12} &= \frac{2\hat{v}P_{22}}{r(t-\delta t)} - \frac{\hat{v}\,P_{11}}{r(t-\delta t)} - \frac{\hat{w}\,P_{12}}{r(t-\delta t)} - 2W1\,(P_{11}P_{12}+P_{12}P_{22}) + \frac{\sin\theta\cos\theta}{W2} \\
\dot{P}_{21} &= \frac{2\hat{v}P_{22}}{r(t-\delta t)} - \frac{\hat{v}\,P_{11}}{r(t-\delta t)} - \frac{\hat{w}\,P_{21}}{r(t-\delta t)} - 2W1(P_{11}P_{21}+P_{21}P_{22}) + \frac{1}{2}\frac{\sin\theta\cos\theta}{W2} \\
\dot{P}_{22} &= -\frac{\hat{v}}{r(t-\delta t)}(P_{12}+P_{21}) - \frac{2\hat{w}}{r(t-\delta t)}P_{22} - 2W1(P_{12}P_{21}+P_{22}^2) + \frac{1}{2}\frac{\cos^2\theta}{W2}\,.
\end{aligned} \tag{5.29}$$

Since $P_{12} = P_{21}$ only five equations must be solved. For this problem $W1 = 1.0$, $W2 = 1.0$, $\hat{w}(0) = w(0) = 0.0$, $\hat{v}(0) = v(0) = 1.0$, and $P(0)$ is chosen to be the null matrix.

The results for this scheme are compared with the optimal results in Figure 7, 8, 9, and 10. Although the thrust angle, θ, shown in Figure 8 appears to be considerably in error during the

middle portion of the flight, this is not true since the thrust itself is very small over this range, and the system is only attempting to remain on the desired trajectory. The suboptimal closed-loop system reaches the desired value at the final time, but at a final mass of 0.8536 rather than the open-loop value of 0.8595. In the suboptimal solution, no inequality thrust magnitude constraints are present, and the maximum thrust used is 0.07786 which compares very favorably with the value of 0.075 specified as an inequality constraint for the open-loop solution.

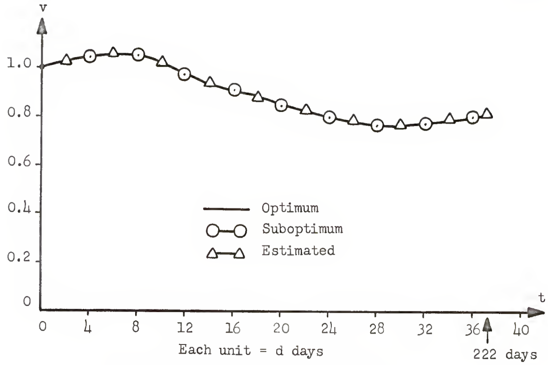

Figure 7. Circumferential velocity

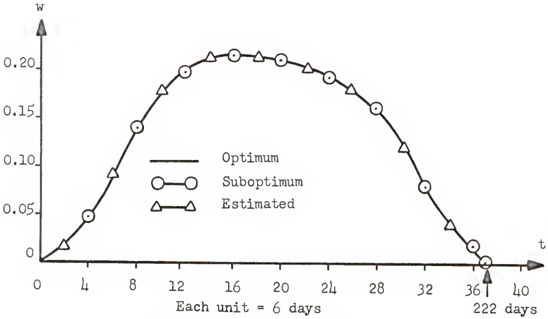

Figure 8. Radial velocity

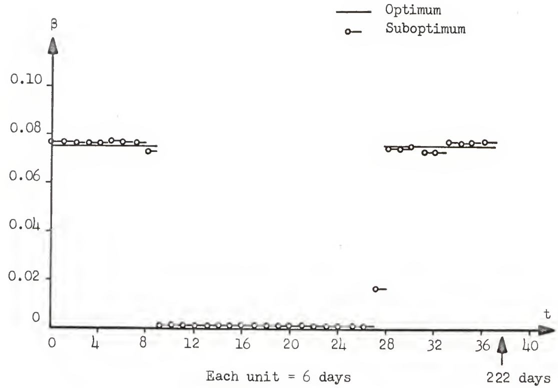

Figure 9. Propellant mass flow, or thrust β

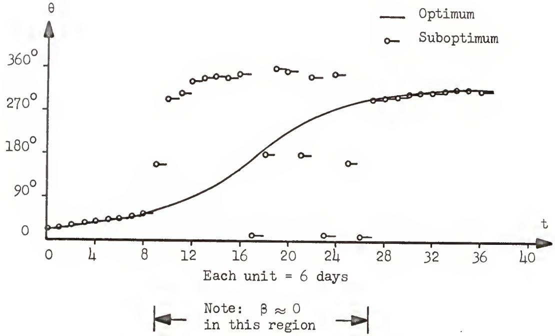

Figure 10. Thrust angle, θ

Example 3: Nuclear Rocket Engine Startup

Rapid startup of a nuclear rocket engine is necessary to conserve propellant, and to reduce the complexity of attitude control problems [3]. For a rapid startup, however, a nuclear rocket engine is difficult to control, particularly since there is disagreement as to the system dynamics, and also the effect of gravity upon flow and heat transfer is not completely determined. For these reasons, it appears that perhaps an on-line method of control is necessary.

Consider the normalized system dynamics as given by Smith and Stenning [4],

$$\begin{aligned}
\dot{n} &= 50\,(\rho n - n + c) \\
\dot{c} &= 0.1\,(n - c) \\
\dot{T} &= 1.471\,(n - PT^{1/2}) \\
\dot{P} &= 0.4\,\left(0.915\,PT^{1/2} - \frac{P^2}{T^{1/2}}\right)
\end{aligned} \tag{5.30}$$

$$\rho = \rho_c - T + 18\,\frac{P}{T} \tag{5.31}$$

where

n = neutron flux density

c = precursor density

T = maximum core surface temperature

P = core inlet stagnation pressure

ρ = reactivity

ρ_c = control poison reactivity.

It is possible to model (5.30) at each $t = t_i$ with a linear model of the form

$$\dot{n} = -50n + 50c + 50n(t_i)\,\rho$$

$$\dot{c} = 0.1n - 0.1c$$

$$\dot{T} = 1.471n - 1.471T^{1/2}(t_i) \qquad (5.32)$$

$$\dot{P} = 0.4\left[\,0.915\,T^{1/2}(t_i) - \frac{P(t_i)}{T^{1/2}(t_i)}\,\right]P\,.$$

It is desired to find a value of reactivity, ρ, at $t = t_i$ which will minimize the cost function

$$V_i = \frac{1}{2}\,\alpha\,[T(t_{i+1}) - T_d(t_{i+1})]^2 + \frac{1}{2}\int_{t_i}^{t_{i+1}} \rho^2\,dt \qquad (5.33)$$

where the values for T_d are taken from a desired trajectory.

Following the formulation given in Chapter IV, the canonic equations are obtained from the maximum principle, yielding (5.32) along with

$$
\begin{aligned}
\dot{\rho} &= -5n(t_i)\,\sigma_1 - 73.55n(t_i)\,\sigma_2 + 50\,\rho \\
\dot{\sigma}_1 &= 0.1\,\sigma_1 - \frac{1}{n(t_i)}\,\rho \\
\dot{\sigma}_2 &= 0 \\
\dot{\sigma}_3 &= 1.471\,T^{1/2}(t_i)\,\sigma_2 - 0.4\left[0.915\,T^{1/2}(t_i) - \frac{P(t_i)}{T^{1/2}(t_i)}\right]\sigma_3 .
\end{aligned}
\tag{5.34}
$$

At $t = t_i$, n, c, T, and P are known to be

$$
\begin{aligned}
&n\ (t_i) \\
&c\ (t_i) \\
&T\ (t_i) \\
&P\ (t_i)
\end{aligned}
\tag{5.35}
$$

and at $t = t_{i+1}$,

$$
\begin{aligned}
\rho\ (t_{i+1}) &= 0 \\
\sigma_1\ (t_{i+1}) &= 0 \\
\sigma_3\ (t_{i+1}) &= 0\ .
\end{aligned}
\tag{5.36}
$$

The transversality condition (4.20) yields

$$\sigma_2(t_{i+1}) = -\ \alpha\,[T(t_{i+1}) - T_d(t_{i+1})]. \tag{5.37}$$

Using the matrix exponential form of solution, given by (4.21), (4.22), and (4.23), $\rho(t_i)$ can be found. Then, $\rho_c(t_i)$, which corresponds to control rod movement, can be found using (5.31).

This procedure is utilized in tracking several desired temperature trajectories given in Figure 11. The corresponding curves for the control poison reactivity are given in Figure 12. The sampling interval, $\Delta t = t_{i+1} - t_i$, varies from 0.01 seconds to 0.05 seconds depending upon the accuracy desired and the particular case. Note that the use of this method allows the generation of the control necessary to maintain a steady-state condition. It can be shown that eventually the system will reach steady-state values of

$$
\begin{aligned}
n_{ss} &= 0.915\, T_{dss}^{3/2} \\
c_{ss} &= n_{ss} \\
P_{ss} &= 0.915\, T_{dss} \\
\rho_{ss} &= 0
\end{aligned}
\tag{5.38}
$$

and

$$\rho_{css} = T_{dss}^{1/2} - 16.47$$

where the subscript ss denotes steady-state and T_{dss} denotes the desired steady-state value for the temperature.

In Figure 13, an optimum temperature trajectory is tracked using a sampling interval of 0.025 seconds. The optimum trajectory results from the control of the system (5.30) over the time interval $t\epsilon(0,10)$ while specifying $T(10) = 0.4$ and minimizing the cost function

$$J = \frac{1}{2}\int_0^{10} \rho^2 \, dt. \tag{5.39}$$

The other state variables, n, c, and P are given in Figure 14, with the total reactivity, ρ, and the control poison reactivity, ρ_c, being given in Figures 15 and 16, respectively.

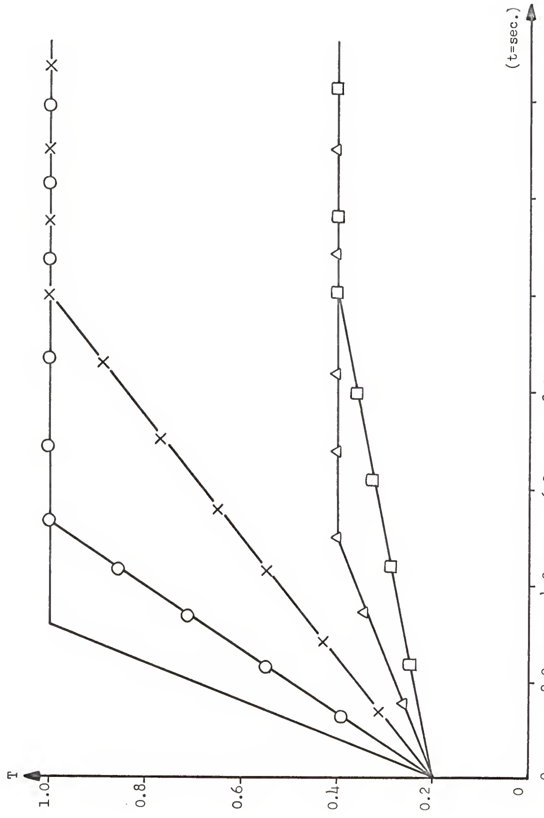

Figure 11. Desired temperature trajectories

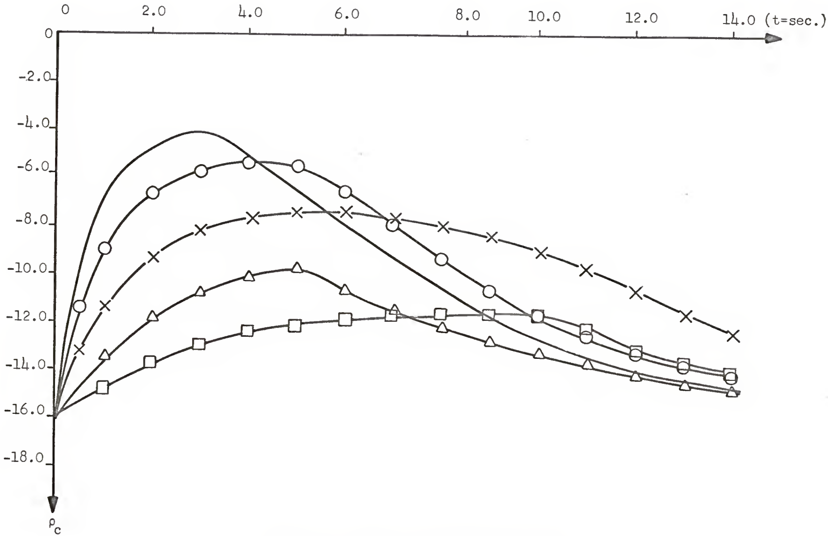

Figure 12. Control poison reactivity

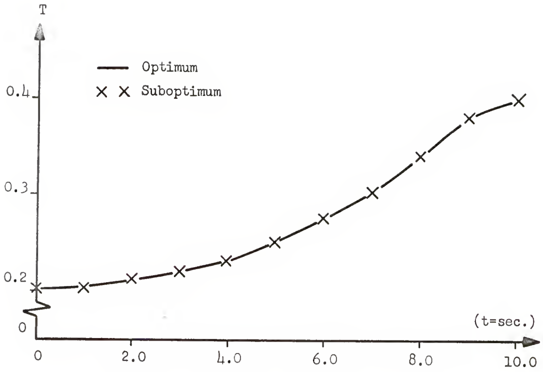

Figure 13. Optimum temperature trajectory

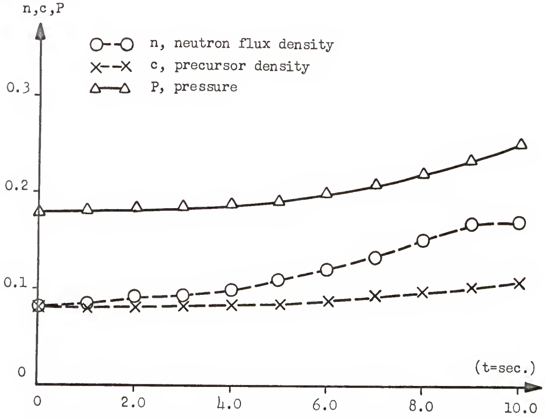

Figure 14. State variables for optimum temperature trajectory

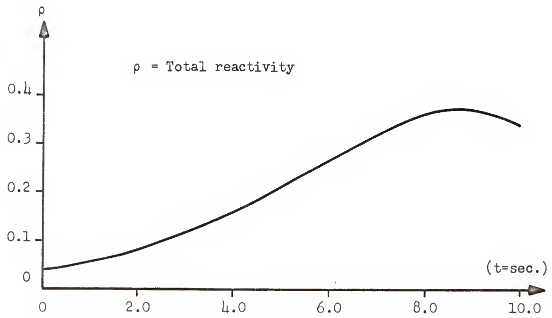

Figure 15. Total reactivity for optimum temperature trajectory

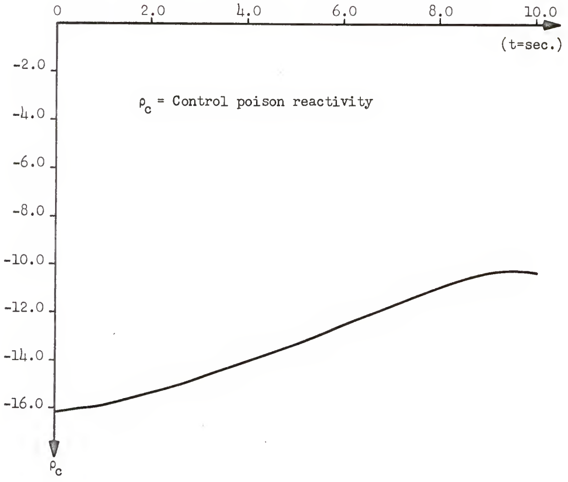

Figure 16. Control poison reactivity for optimum temperature trajectory

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3. Humphries, J. T., R. E. Uhrig, and A. P. Sage, "Adaptive Control Concepts for Nuclear Rocket Propulsion Systems," American Institute of Aeronautics and Astronautics Second Propulsion Joint Specialist Conference, June, 1966.

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CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

The results presented in this dissertation constitute the development of two methods of control for continuous nonlinear systems which appear feasible for use in an on-line fashion. These methods of control are combined with a method of state and parameter estimation to yield an adaptive system which attempts to operate in some optimal fashion or to track an acceptable trajectory. The emphasis in the derivation of these control laws is placed on the simplicity of the formulation and the speed and ease at which the control can be calculated and applied.

The development of the suboptimal adaptive regulator scheme presented in Chapter III utilizes the identification of a linear constant-coefficient system at discrete instants of time and the calculation of a control which minimizes a time weighted quadratic performance index over the remaining time to go. Since this method allows the system to adapt to new trajectories, it is possible to reduce the cost over that which would result from the application of an open-loop control, based on dynamics that are not completely known a priori.

The suboptimal adaptive trajectory control scheme presented in Chapter IV is also based on the identification of a linear

constant-coefficient model at discrete instants of time. The control is then calculated in such a way as to force the system to track some predetermined desired trajectory. This usually insures an acceptable response even though it may not be optimal.

Applications of these two methods of control are given in Chapter V. The problems considered are nuclear reactor startup, low thrust orbital transfer, and nuclear rocket engine startup. It is difficult to present a general approach which is applicable to all systems. In each instance, there must be some previous knowledge of the system, and preliminary decisions must be made concerning the type of model to be used, sampling interval size, weighting factors, etc. These decisions must be based on trial and error procedures along with a good understanding of the results to be obtained.

During the development of the material presented in this dissertation, many problems were encountered which deserve further investigation. Among these were:

1. Stability problem. The utilization of these methods of control can present certain stability problems which cannot be readily analyzed. These problems are usually the result of either inaccurate system models, large sampling intervals, or both. For example, when using the method of trajectory control, the system response in certain cases can be made to oscillate and eventually go unstable if the sampling interval is too large. This, of course, results from the system continuously overcompensating. Although this problem was evident in the examples used in Chapter V, it was successfully eliminated

by choosing smaller sampling intervals. More investigation is necessary to determine the relationship between instability, sampling interval size, and model errors.

2. Prediction of system dynamics. In each of the methods of control presented, a model was chosen at $t = t_i$ based on information at $t = t_i$. However, no use was made of the past history of the system to possibly obtain a better estimate of the system dynamics over the next period of interest. To be more specific, perhaps a method of learning, averaging, or extrapolation, could be used. This would be particularly advantageous for the regulator method of control where the use of a linear constant-coefficient system is generally less accurate than for the trajectory method of control.

3. Minimization of storage space for trajectory method. Since storage space is often excessive for trajectory methods, more investigation should be conducted to aid in the reduction of storage requirements. One possibility would be the use of small sampling intervals when the system is responding quickly and large sampling intervals otherwise [1].

4. Search method for finding optimal controls. It is, in general, very difficult to find optimal controls for nonlinear systems. However, in some instances, an approximation of the optimal control can be found off-line by using either of the methods of control presented in this dissertation. This involves the use of different weighting factors in the regulator method, and different desired trajectories in the trajectory method. Although this is trial and error, it may be

possible to combine these methods with existing methods, such as gradient techniques, in order to attain a more sophisticated search method.

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APPENDIX A

DERIVATION OF THE INVARIENT IMBEDDING EQUATION

For the derivation of the invarient imbedding equation consider the two-point boundary-value problem described by the vector differential equations

$$\dot{\underline{x}} = \underline{f}\ (\underline{x},\ \underline{y},\ t)$$
$$\dot{\underline{y}} = \underline{g}\ (\underline{x},\ \underline{y},\ t) \qquad (A.1)$$

where $\underline{x}$ and $\underline{y}$ are n-dimensional vectors. The boundary conditions are given by

$$\underline{y}(0) = a$$
$$\underline{y}(T) = \underline{b} \qquad (A.2)$$

with the process starting at t = 0 and ending at t = T. Let

$$\underline{x}(T) = \underline{r}\ (\underline{C},\ T) \qquad (A.3)$$

where

$$\underline{y}(T) = \underline{C}. \qquad (A.4)$$

With $\underline{C}$ and T regarded as independent variables, write

$$\underline{r}\ (\underline{C} + \underline{\Delta C},\ T + \Delta T) = \underline{r}\ (\underline{C},\ T) + \underline{f}\ (\underline{r},\ \underline{C},\ T)\ \Delta T + O(\Delta^2) \qquad (A.5)$$

where

$$\lim_{\Delta \to 0} \frac{O(\Delta^2)}{\Delta} = 0 \ . \tag{A.6}$$

The left side of (A.4) can be expanded in a Taylor series to yiwld

$$\underline{r}\ (\underline{C} + \underline{\Delta C},\ T + \Delta T = \underline{r}\ (\underline{C},\ T) + \frac{\partial r}{\partial \underline{C}}\ \Delta C + \frac{\partial r}{\partial T}\ \Delta T + O(\Delta^2). \tag{A.7}$$

From (A.1) and (A.4) write

$$\underline{\Delta C} = \underline{g}\ (\underline{r},\ \underline{C},\ T) + O(\Delta^2). \tag{A.8}$$

Then, by equating the right-hand sides of equations (A.6) and (A.7), and substituting (A.8) for $\underline{\Delta C}$, the equation

$$\frac{\partial r}{\partial T} + \frac{\partial r}{\partial \underline{C}}\ \underline{g}\ (\underline{r},\ \underline{C},\ T) = \underline{f}\ (\underline{r},\ \underline{C},\ T) \tag{A.9}$$

results. This is a partial differential equation which, with the proper conditions on $\underline{r}$, governs the dependence of the missing terminal conditions on $\underline{x}$ as a function of the duration of the process and the terminal conditions on $\underline{y}$.

APPENDIX B

DERIVATION OF SEQUENTIAL ESTIMATOR EQUATIONS

Consider the class of systems defined by

$$\dot{\underline{x}} = \underline{g}\ (t,\ \underline{x}) + k\ (t,\ \underline{x})\ \underline{w}$$
$$\underline{z} = \underline{h}\ (t,\ \underline{x}) + \underline{v} \tag{B.1}$$

where

$\underline{g}\ (t,\ \underline{x})$ = n vector function

$k\ (t,\ \underline{x})$ = n x p vector function

$\underline{w}$ = p vector unknown input

$\underline{h}\ (t,\ \underline{x})$ = m vector function

$\underline{z}$ = m vector output

$\underline{v}$ = m vector measurement error.

It is desired to find the least square estimate of $\underline{x}$, designated $\hat{\underline{x}}$, which minimizes the cost function

$$J' = \int_o^T \left[\| \underline{z} - \underline{h}(t,\underline{x}) \|^2_{W1} + \| \dot{\underline{x}} - \underline{g}(t,\underline{x}) \|^2_{W2} \right] dt \tag{B.2}$$

or, alternately written as

$$J' = \int_o^T \left[\| \underline{v} \|^2_{W1} + \| \underline{w} \|^2_{k'W2k} \right] dt \tag{B.3}$$

where W1 and W2 are weighting matrices which determine the relative weighting to be placed on the individual terms of the cost function.

Utilizing the maximum principle, the Hamiltonian is written as

$$H = \| \underline{z} - \underline{h}(t,\underline{x}) \|^2_{W1} + \| \underline{w} \|^2_{b} + \underline{\lambda}'[\underline{g}(t,x) + K(t,x)\ \underline{w}] \tag{B.4}$$

where v = k'W2k. The canonic equations are given by

$$\begin{aligned} \dot{\underline{x}} &= \frac{\partial H}{\partial \underline{\lambda}} \\ \dot{\underline{\lambda}} &= -\frac{\partial H}{\partial \underline{x}} \end{aligned} \tag{B.5}$$

with

$$\frac{\partial H}{\partial \underline{w}} = 0. \tag{B.6}$$

Since T is fixed, x(0) and x(T) are free, therefore

$$\begin{aligned} \underline{\lambda}(0) &= \underline{0} \\ \underline{\lambda}(T) &= \underline{0}. \end{aligned} \tag{B.7}$$

It is necessary to solve the two-point boundary-value problem given by (B.5) and (B.7). Consider the more general problem of letting

$$\begin{aligned} \underline{\lambda}(0) &= 0 \\ \underline{\lambda}(T) &= C \end{aligned} \tag{B.8}$$

and $\underline{x}(T) = \underline{r}(\underline{C},T)$. Then, $\underline{r}(\underline{C},T)$ satisfies the invarient imbedding equations

$$\frac{\partial \underline{r}}{\partial T} - \frac{\partial \underline{r}}{\partial \underline{C}} \frac{\partial H}{\partial \underline{r}} (\underline{r},\underline{C},T) = + \frac{\partial H}{\partial \underline{C}} (\underline{r},\underline{C},T) . \tag{B.9}$$

Assume a solution of the form

$$\underline{r}(\underline{C},T) = \underline{\hat{x}}(T) + P(T)\,\underline{C} \tag{B.10}$$

where P(T) is an n x n matrix. Substitute (B.10) into (B.9) and expand the result about $\underline{r}(0,T)$. After collecting terms of order $\underline{C}^0$, $\underline{C}^1$, $\underline{C}^2$, and $\underline{C}^3$, equations for $\dot{\underline{x}}$ and $\dot{P}$ can be found. With the $\dot{P}$ equation divided by $\underline{C}$ and with P replaced by -P, it is noted that only those solutions for which $\underline{C} = \underline{0}$ are of interest. Thus the sequential estimator equations become

$$\dot{\underline{\hat{x}}} = \underline{g}(T,\underline{\hat{x}}) + 2P(T)\,H\,(T,\hat{x})\,W1\,[\underline{z}(T) - \underline{h}(T,\underline{\hat{x}})]$$

$$\dot{P} = g_{\underline{\hat{x}}}(T,\underline{\hat{x}}) + P_{g'_{\hat{x}}}(T,\underline{\hat{x}}) + 2P\,[H\,W1\,\{\underline{z}(T) - \underline{h}(T,\underline{\hat{x}})\}]_{\underline{\hat{x}}}\,P$$
$$+ \frac{1}{2}\,\underline{k}(T,\underline{\hat{x}})\,V^{-1}(T,\underline{\hat{x}})\,k'(T,\underline{\hat{x}}) \tag{B.11}$$

where

$$g_{\underline{\hat{x}}} = \frac{\partial \underline{g}}{\partial \underline{\hat{x}}}$$

$$H = \left[\frac{\partial \underline{h}}{\partial \underline{x}}\right]'$$

and $[H\,W1\,\{\underline{z}(t) - \underline{h}(T,\underline{x})\}]_{\underline{\hat{x}}}$ is an n x n matrix.

BIOGRAPHICAL SKETCH

Thomas Walter Ellis was born December 1, 1939, in Memphis, Tennessee. In June, 1957, he was graduated from Central High School. In June, 1962, he received the degree of Bachelor of Science from Christian Brothers College, Memphis, Tennessee. The following fall he entered the University of Florida and worked as a teaching assistant until August, 1963, when he received the degree of Master of Electrical Engineering. The following academic year he taught in the Department of Electrical Engineering at Christian Brothers College. During the summer of 1964, he worked as a communications engineer with Pan American World Airways at Cape Kennedy. That fall he returned to the University of Florida to pursue work toward the degree of Doctor of Philosophy and was awarded a Graduate School Fellowship. Since September, 1965, his work has been sponsored by a National Aeronautics and Space Administration Traineeship.

Thomas Walter Ellis is married to the former Ann Marie Militana. He is a member of the Institute of Electrical and Electronic Engineers and Simulation Councils, Inc.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Engineering and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

December 17, 1966

J. G. Rathers

Dean, College of Engineering

Dean, Graduate School

Supervisory Committee:

A P Sage

Chairman

J. R. O'Malley